Journal of Applied Mechanics and Technical Physics, Vol. 42, No. 1, pp. 140–145, 2001

INDENTATION OF A RIGID BODY INTO AN ELASTIC PLATE

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UDC 539.3

The problem of a punch shaped like an elliptic paraboloid pressed into an elastic plate is studied under the assumption that the contact region is small. The action of the punch on the plate is modeled by point forces and moments. The method of joined asymptotic expansions is used to formulate the problem of one-sided contact for the internal asymptotic representation; the problem is solved with the use of the results obtained by L. A. Galin. The coordinates of the center of the elliptic contact region, its dimensions, and the angle of rotation are determined. The moments which ensure translational indentation of the punch are calculated and an equation that relates displacements of the punch to the force acting on it is given.

1. Formulation of the Problem. Let a punch shaped like an elliptic paraboloid

$$\Phi(\boldsymbol{x}^0;\boldsymbol{x}) = (2R_1)^{-1}(x_1 - x_1^0)^2 + (2R_2)^{-1}(x_2 - x_2^0)^2 \tag{1.1}$$

be pressed into an elastic plate Ω with the flexural rigidity D which is fixed along the edge $\partial\Omega$. Here R_1 and R_2 are the radii of curvature of the principal normal sections at the vertex of the punch $\boldsymbol{x}^0 \in \Omega$. We denote the translational displacement of the punch by δ_0.

The deflection of the plate is determined from the solution of the problem (see, e.g., [1])

$$u(\boldsymbol{x}) > \delta_0 - \Phi(\boldsymbol{x}^0;\boldsymbol{x}) \quad \Longrightarrow \quad D\Delta_x\Delta_x u(\boldsymbol{x}) = 0; \tag{1.2}$$

$$u(\boldsymbol{x}) = \delta_0 - \Phi(\boldsymbol{x}^0;\boldsymbol{x}) \quad \Longrightarrow \quad D\Delta_x\Delta_x u(\boldsymbol{x}) \geqslant 0; \tag{1.3}$$

$$u(\boldsymbol{x}) \geqslant \delta_0 - \Phi(\boldsymbol{x}^0;\boldsymbol{x}), \qquad \boldsymbol{x} = (x_1, x_2) \in \Omega; \tag{1.4}$$

$$u(\boldsymbol{x}) = 0, \qquad \partial_n u(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \partial\Omega. \tag{1.5}$$

The contact region Σ, where equality (1.3) holds, is unknown *a priori.* In accordance with the assumed shape of the punch (1.1), the pressure $p(x_1, x_2) = -D\Delta_x\Delta_x\Phi(\boldsymbol{x}^0;\boldsymbol{x})$ exerted by the punch on the plate is concentrated on the contact-region contour $\partial\Sigma$.

We study problem (1.2)–(1.5) under the assumption that the region Σ is small. Evidently, its dimensions are "controlled" by the parameters δ_0, R_1, and R_2. Denoting the small positive parameter by ε, we set

$$R_1 = \varepsilon R_1^*, \qquad R_2 = \varepsilon R_2^*, \qquad \delta_0 = \varepsilon\delta_0^*, \tag{1.6}$$

where the quantities δ_0^*, R_1^*, and R_2^* are comparable with the distance d_0 from the point $\boldsymbol{x}^0$ to the boundary $\partial\Omega$.

The above problem with one-sided constraints and related problems have been studied within the framework of the theory of variational inequalities (see, e.g., [1, 2]). Glowinski, Lions, and Tremolieres [3] and Kovtunenko [4] proposed numerical algorithms of solution. Nazarov [5] developed asymptotic methods

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Makarov State Marine Academy, St. Petersburg 199106. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 42, No. 1, pp. 157–163, January–February, 2001. Original article submitted October 11, 1999.

 0021-8944/01/4201-0140 $25.00

to study variational inequalities. Khludnev [6] considered optimal-control problems. Galin [7] obtained an approximate solution of the problem of a punch (1.1) pressed into a circular plate under the assumption that plate deflection at a distance from the contact region is described by the solution of the problem of a plate with a point force applied at its center. Rosenberg [8] and Grigolyuk and Tolkachev [9] studied the axisymmetric problem. Cherepanov [10] considered the contact problem for a simply supported polygonal plate.

If the punch vertex does not coincide with the center of the plate, certain moments should be applied to the punch to ensure its translational displacement. The aim of the present study is to approximate these moments and generalize the results of [7]. Problem (1.1)–(1.5) is solved by the method of joined asymptotic expansions [11]. The method outlined in [12] is used to formulate the problem of one-sided contact for the boundary layer whose solution is given by the formulas of [7]. Campbell and Nazarov [13] applied the method of joined asymptotic expansions to study vibrations of an elastic plate having a small rigid inclusion with a specified law of motion.

2. External and Internal Asymptotic Representations. We denote the solution of the problem of bending of a plate Ω loaded by a point force at the point $\boldsymbol{x}^0$ by $\Gamma(\boldsymbol{x}^0;\boldsymbol{x})$:

$$\Gamma(\boldsymbol{x}^0;\boldsymbol{x}) = \frac{1}{8\pi D}\,|\boldsymbol{x}-\boldsymbol{x}^0|^2 \ln\frac{|\boldsymbol{x}-\boldsymbol{x}^0|}{r_0} + \gamma(\boldsymbol{x}^0;\boldsymbol{x}). \tag{2.1}$$

Here r_0 is a constant having the dimension of length and

$$\gamma(\boldsymbol{x}^0;\boldsymbol{x}) = \gamma(\boldsymbol{x}^0;\boldsymbol{x}^0) - \gamma_2(\boldsymbol{x}^0)(x_1-x_1^0) + \gamma_1(\boldsymbol{x}^0)(x_2-x_2^0)$$

$$+ \sum_{i,j=1}^{2}\gamma_{ij}(\boldsymbol{x}^0)(x_i-x_i^0)(x_j-x_j^0) + O(|\boldsymbol{x}-\boldsymbol{x}^0|^3), \quad \boldsymbol{x}\to\boldsymbol{x}^0. \tag{2.2}$$

Remark 2.1. If $G(\boldsymbol{x}^0;\boldsymbol{x})$ is Green's harmonic function for the Dirichlet problem, then $G(\boldsymbol{x}^0;\boldsymbol{x}) = -(2\pi)^{-1}\ln(|\boldsymbol{x}-\boldsymbol{x}^0|/r_0) + o(1)$ as $\boldsymbol{x}\to\boldsymbol{x}^0$. In the case of a simply connected region Ω, the parameter r_0 is the internal conformal radius of the region Ω relative to the point $\boldsymbol{x}^0$ (see, e.g., [14]). One can show (see [14, Problem No. 122]) that $r_0 \geqslant d_0$. The Green's biharmonic function (2.1) can be written in the form $\Gamma(\boldsymbol{x}^0;\boldsymbol{x}) = -(4D)^{-1}|\boldsymbol{x}-\boldsymbol{x}^0|^2 G(\boldsymbol{x}^0;\boldsymbol{x}) + \tilde\gamma(\boldsymbol{x}^0;\boldsymbol{x})$, where $\tilde\gamma$ is a regular function. We note that the quantity $4\sqrt{\pi D\gamma(\boldsymbol{x}^0;\boldsymbol{x}^0)}$ is interpreted as an internal biharmonic radius.

We write the solutions of the problem of a plate Ω loaded by point moments at the point $\boldsymbol{x}^0$

$$\Gamma^{(1)}(\boldsymbol{x}^0;\boldsymbol{x}) = -\frac{1}{4\pi D}\,(x_2-x_2^0)\ln\frac{|\boldsymbol{x}-\boldsymbol{x}^0|}{r_0} + \gamma^{(1)}(\boldsymbol{x}^0;\boldsymbol{x}); \tag{2.3}$$

$$\Gamma^{(2)}(\boldsymbol{x}^0;\boldsymbol{x}) = \frac{1}{4\pi D}\,(x_1-x_1^0)\ln\frac{|\boldsymbol{x}-\boldsymbol{x}^0|}{r_0} + \gamma^{(2)}(\boldsymbol{x}^0;\boldsymbol{x}). \tag{2.4}$$

For regular parts of the functions (2.3) and (2.4), the formula

$$\gamma^{(i)}(\boldsymbol{x}^0;\boldsymbol{x}) = \gamma^{(i)}(\boldsymbol{x}^0;\boldsymbol{x}^0) + O(|\boldsymbol{x}-\boldsymbol{x}^0|), \quad \boldsymbol{x}\to\boldsymbol{x}^0 \qquad (i=1,2) \tag{2.5}$$

is valid.

At a distance from the contact region, the action of the punch on the elastic plate is modeled by the reactions concentrated at the point $\boldsymbol{x}^0$:

$$v(\boldsymbol{x}) = P\Gamma(\boldsymbol{x}^0;\boldsymbol{x}) + \sum_{i=1}^{2} M_i\Gamma^{(i)}(\boldsymbol{x}^0;\boldsymbol{x}). \tag{2.6}$$

As a first approximation, we have $P\gamma(\boldsymbol{x}^0;\boldsymbol{x}^0) = \delta_0$; therefore, according to (1.6), we set

$$P = \varepsilon P^*. \tag{2.7}$$

In the neighborhood of the contact region $\Sigma(\varepsilon)$, we use the "extended" coordinates

$$\boldsymbol{\xi} = (\xi_1, \xi_2), \qquad \xi_i = \varepsilon^{-1}(x_i - x_i^0). \tag{2.8}$$

According to (1.2)–(1.4) and (2.8), the internal asymptotic representation of the solution of the initial problem $w(\varepsilon; \boldsymbol{\xi})$ satisfies the relations

$$w(\varepsilon; \boldsymbol{\xi}) > \varepsilon(\delta_0^* - \Phi^*(\boldsymbol{\xi})) \implies \Delta_\xi \Delta_\xi w(\varepsilon; \boldsymbol{\xi}) = 0; \tag{2.9}$$

$$w(\varepsilon; \boldsymbol{\xi}) = \varepsilon(\delta_0^* - \Phi^*(\boldsymbol{\xi})) \implies \Delta_\xi \Delta_\xi w(\varepsilon; \boldsymbol{\xi}) \geqslant 0; \tag{2.10}$$

$$w(\varepsilon; \boldsymbol{\xi}) \geqslant \varepsilon(\delta_0^* - \Phi^*(\boldsymbol{\xi})), \quad \boldsymbol{\xi} \in \mathbb{R}^2; \qquad \Phi^*(\boldsymbol{\xi}) = (2R_1^*)^{-1}\xi_1^2 + (2R_2^*)^{-1}\xi_2^2. \tag{2.11}$$

Since the scale is changed, the distance from the punch vertex to the plate edge becomes equal to $\varepsilon^{-1}d_0$ and, hence, for small ε, formulas (2.9) and (2.10) are valid on the entire plane. The discarded boundary conditions (1.5) are replaced by the asymptotic conditions for $w(\varepsilon; \boldsymbol{\xi})$ as $\boldsymbol{\xi} \to \infty$.

Bearing in mind (2.1)–(2.5), (2.7), and (2.8), we obtain the following expansion for the function (2.6):

$$v(\boldsymbol{x}^0 + \varepsilon\boldsymbol{\xi}) = \varepsilon P^* \Big\{ (8\pi D)^{-1} \varepsilon^2 |\boldsymbol{\xi}|^2 \ln \frac{\varepsilon|\boldsymbol{\xi}|}{r_0} + \gamma(\boldsymbol{x}^0; \boldsymbol{x}^0) + \varepsilon[-\gamma_2(\boldsymbol{x}^0)\xi_1 + \gamma_1(\boldsymbol{x}^0)\xi_2]$$

$$+ \varepsilon^2 \sum_{i,j=1}^{2} \gamma_{ij}(\boldsymbol{x}^0)\xi_i\xi_j \Big\} + \sum_{i=1}^{2} M_i \gamma^{(i)}(\boldsymbol{x}^0; \boldsymbol{x}^0) + \varepsilon(4\pi D)^{-1}(M_2\xi_1 - M_1\xi_2) \ln \frac{\varepsilon|\boldsymbol{\xi}|}{r_0} + \ldots . \tag{2.12}$$

Here the omission points denote insignificant (in subsequent calculations) terms. It is noteworthy that as $\varepsilon \to 0$, one cannot determine *a priori* the order of the moments M_i. We assume that

$$M_i = \varepsilon^2 M_i^* \qquad (i = 1,\, 2). \tag{2.13}$$

Below, we show that $M_i = O(\varepsilon^3)$; therefore, in the derivation of formula (2.12), we ignore terms of the orders $O(\varepsilon^3|\boldsymbol{\xi}|^3)$ and $O(\varepsilon^3|\boldsymbol{\xi}|)$ compared to unity [see, in particular, the braced expression in (2.12)].

With the use of the method of joined asymptotic expansions, relation (2.12) allows one to formulate the condition for the boundary layer at infinity which we seek in the form

$$w(\varepsilon; \boldsymbol{\xi}) = \varepsilon[V^*(\boldsymbol{\xi}) + W(\varepsilon; \boldsymbol{\xi})]; \tag{2.14}$$

$$V^*(\boldsymbol{\xi}) = P^* \Big\{ \gamma(\boldsymbol{x}^0; \boldsymbol{x}^0) + \varepsilon[-\gamma_2(\boldsymbol{x}^0)\xi_1 + \gamma_1(\boldsymbol{x}^0)\xi_2] + \varepsilon^2 \sum_{i,j=1}^{2} \gamma_{ij}(\boldsymbol{x}^0)\xi_i\xi_j \Big\} + \varepsilon \sum_{i=1}^{2} M_i^* \gamma^{(i)}(\boldsymbol{x}^0; \boldsymbol{x}^0). \tag{2.15}$$

As $|\boldsymbol{\xi}| \to \infty$, (2.12) yields the following representation for W:

$$W(\varepsilon; \boldsymbol{\xi}) = \varepsilon^2 \Big[\frac{P^*}{8\pi D} |\boldsymbol{\xi}|^2 \ln \frac{\varepsilon|\boldsymbol{\xi}|}{r_0} + \frac{1}{4\pi D} (M_2^*\xi_1 - M_1^*\xi_2) \ln \frac{\varepsilon|\boldsymbol{\xi}|}{r_0} \Big] + \ldots . \tag{2.16}$$

Relations (2.9)–(2.11) and (2.14)–(2.16) form a model problem of one-sided contact for an infinite plate. We solve this problem using the results of [7].

3. Determination of the Moments Acting on the Punch. Substituting (2.14) into (2.9)–(2.11), we infer that the function W given by (2.16) satisfies the relations

$$W(\varepsilon; \boldsymbol{\xi}) > \delta_0^* - \Phi^*(\boldsymbol{\xi}) - V^*(\boldsymbol{\xi}) \implies \Delta_\xi \Delta_\xi W(\varepsilon; \boldsymbol{\xi}) = 0; \tag{3.1}$$

$$W(\varepsilon; \boldsymbol{\xi}) = \delta_0^* - \Phi^*(\boldsymbol{\xi}) - V^*(\boldsymbol{\xi}) \implies \Delta_\xi \Delta_\xi W(\varepsilon; \boldsymbol{\xi}) \geqslant 0; \tag{3.2}$$

$$W(\varepsilon; \boldsymbol{\xi}) \geqslant \delta_0^* - \Phi^*(\boldsymbol{\xi}) - V^*(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \mathbb{R}^2. \tag{3.3}$$

We assume that $\gamma_{12}(\boldsymbol{x}^0) = \gamma_{21}(\boldsymbol{x}^0) = 0$. Separating the complete squares, we obtain

$$\delta_0^* - \Phi^*(\boldsymbol{\xi}) - V^*(\boldsymbol{\xi}) = O_0^* - \sum_{i=1}^{2} \Big[\frac{1}{2R_i^*} + \varepsilon^2 P^* \gamma_{ii}(\boldsymbol{x}^0) \Big] (\xi_i - \xi_i^c)^2; \tag{3.4}$$

$$O_0^* = \delta_0^* - P^*\gamma(\boldsymbol{x}^0; \boldsymbol{x}^0) - \varepsilon \sum_{i=1}^{2} M_i^* \gamma^{(i)}(\boldsymbol{x}^0; \boldsymbol{x}^0) + \sum_{i=1}^{2} (2R_i^*)^{-1} (\xi_i^c)^2,$$
$$\xi_1^c = \varepsilon P^* R_1^* \gamma_2(\boldsymbol{x}^0), \qquad \xi_2^c = -\varepsilon P^* R_2^* \gamma_1(\boldsymbol{x}^0). \tag{3.5}$$

In (3.4), terms of the order $O(\varepsilon^3)$ are discarded, since this corresponds to the accuracy of formulas (2.12) and (2.15).

In this stage, we determine the moments M_i^*. We note that the behavior of W at infinity is determined by the first, rapidly increasing term in square brackets in (2.16). Hence, according to (3.4), the center of the contact region shifts to the point with coordinates (3.5). Imposing the following constraint on the function W

$$W(\varepsilon; \boldsymbol{\xi}) = \frac{\varepsilon^2 P^*}{8\pi D} |\boldsymbol{\xi} - \boldsymbol{\xi}^c|^2 \ln \frac{|\boldsymbol{\xi} - \boldsymbol{\xi}^c|}{r_0/\varepsilon} + O\Big(\ln \frac{|\boldsymbol{\xi} - \boldsymbol{\xi}^c|}{r_0/\varepsilon}\Big), \qquad |\boldsymbol{\xi} - \boldsymbol{\xi}^c| \to \infty, \tag{3.6}$$

we obtain the equalities

$$M_1^* = P^* \xi_2^c, \qquad M_2^* = -P^* \xi_1^c. \tag{3.7}$$

Bearing in mind (3.5), (3.7), and (2.13), we infer that M_i is of the order $O(\varepsilon^3)$. Finally, reverting to the real scale and using (1.6), (2.7), and (2.8), we find

$$x_1^c = x_1^0 + P R_1 \gamma_2(\boldsymbol{x}^0), \qquad x_2^c = x_2^0 - P R_2 \gamma_1(\boldsymbol{x}^0). \tag{3.8}$$

We use the results of [7] to construct the solution of the model problem (3.1)–(3.3) and (3.6). We introduce the complex variable $z = \xi_1 - \xi_1^c + i(\xi_2 - \xi_2^c)$. Since a second-degree polynomial enters the right side of (3.4), the contact region Σ^* is elliptic and its complement to the enhanced complex plane is the image of the exterior of a unit circle for conformal mapping

$$z = \omega(\zeta), \qquad \omega(\zeta) = c^*(\zeta + m\zeta^{-1}); \tag{3.9}$$

$$c^* = \frac{r_0}{\varepsilon} \exp\Big\{ -\frac{4\pi D}{\varepsilon^2 P^*} \frac{R_1^* + R_2^*}{2R_1^* R_2^*} - 4\pi D[\gamma_{11}(\boldsymbol{x}^0) + \gamma_{22}(\boldsymbol{x}^0)] - 1 \Big\}; \tag{3.10}$$

$$m = \frac{8\pi D}{\varepsilon^2 P^*} \frac{R_1^* - R_2^*}{2R_1^* R_2^*} + 8\pi D[\gamma_{22}(\boldsymbol{x}^0) - \gamma_{11}(\boldsymbol{x}^0)]. \tag{3.11}$$

Using the Goursat formula, we write the internal asymptotic representation (2.14) in the form

$$w(\varepsilon; \boldsymbol{\xi}) = \varepsilon(\delta_0^* - \Phi^*(\boldsymbol{\xi})) + \varepsilon \operatorname{Re}[\bar{z}\varphi(z) + \chi(z)]. \tag{3.12}$$

The derivatives of the complex potentials are given by [7]

$$\varphi'[\omega(\zeta)] = \frac{\varepsilon^2 P^*}{8\pi D} \ln \zeta, \qquad \chi''[\omega(\zeta)] = -\frac{\varepsilon^2 P^*}{8\pi D} \frac{1 + m\zeta^2}{\zeta^2 - m}. \tag{3.13}$$

With allowance for (3.9) and (3.13), the equality

$$\varphi[\omega(\zeta)] = \int \varphi'[\omega(\zeta)] \frac{d\omega(\zeta)}{d\zeta} \, d\zeta$$

becomes

$$\varphi[\omega(\zeta)] = \frac{\varepsilon^2 P^*}{8\pi D} c^* \Big[\Big(\zeta + \frac{m}{\zeta}\Big) \ln \zeta - \zeta + \frac{m}{\zeta} \Big]. \tag{3.14}$$

The second formula (3.13) can be integrated twice to give

$$\chi[\omega(\zeta)] = \frac{\varepsilon^2 P^*}{8\pi D} (c^*)^2 \Big[(1 + m^2) \ln \zeta - \frac{m}{2} \Big(\zeta^2 - \frac{1}{\zeta^2}\Big) \Big] + C. \tag{3.15}$$

The integration constant in (3.15) is determined from the condition that the second term in (3.12) vanishes at the contour $\partial\Sigma^*$ of the contact region (for $|\zeta| = 1$) and has the form $C = \varepsilon^2 P^* (8\pi D)^{-1} (c^*)^2 (1 - m^2)$.

4. Force–Displacement Relation for the Punch. We substitute (3.14) and (3.15) into (3.12) and study the behavior of $w(\varepsilon;\boldsymbol{\xi})$ as $|\boldsymbol{\xi}|\to\infty$. Simple calculations lead to the relation

$$w(\varepsilon;\boldsymbol{\xi})=\varepsilon\delta_0^*-\varepsilon\Big(\frac{\xi_1^2}{2R_1^*}+\frac{\xi_2^2}{2R_2^*}\Big)+\frac{\varepsilon^3P^*}{8\pi D}|\boldsymbol{\xi}-\boldsymbol{\xi}^c|^2\ln\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}^c|}{r_0/\varepsilon}$$

$$+\varepsilon\sum_{i=1}^{2}\Big(\frac{1}{2R_i^*}+\varepsilon^2P^*\gamma_{ii}(\boldsymbol{x}^0)\Big)(\xi_i-\xi_i^c)^2+\frac{\varepsilon^3P^*}{8\pi D}(c^*)^2(1+m^2)\ln\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}^c|}{c^*}$$

$$+\frac{\varepsilon^3P^*}{8\pi D}(c^*)^2m\,\frac{(\xi_1-\xi_1^c)^2-(\xi_2-\xi_2^c)^2}{|\boldsymbol{\xi}-\boldsymbol{\xi}^c|^2}+\frac{\varepsilon^3P^*}{8\pi D}(c^*)^2\Big(1-\frac{m^2}{2}\Big)+O(|\boldsymbol{\xi}|^{-1}),\quad|\boldsymbol{\xi}|\to\infty. \tag{4.1}$$

We compare the expansion (4.1) with (2.12). First, within the framework of the method of joined asymptotic expansions, the presence of the term $\ln(|\boldsymbol{\xi}-\boldsymbol{\xi}^c|/c^*)$ in (4.1) shows that the external asymptotic representation (2.6) must be refined by a corresponding singular solution. However, according to (3.10), the quantity $\varepsilon c^*/r_0$ decreases exponentially as $\varepsilon\to0$. At the same time, since $|\boldsymbol{\xi}-\boldsymbol{\xi}^c|^2=|\boldsymbol{\xi}|^2-2\sum_{i=1}^{2}\xi_i\xi_i^c+|\boldsymbol{\xi}^c|^2$, the expansion of the third term on the right side of (4.1) contains the term $\varepsilon^3P^*(8\pi D)^{-1}\varepsilon^2(P^*)^2[(R_1^*)^2\gamma_2(\boldsymbol{x}^0)^2+(R_2^*)^2\gamma_1(\boldsymbol{x}^0)^2]\ln(\varepsilon|\boldsymbol{\xi}|/r_0)$.

Thus, the above singular solution of the biharmonic equation with a logarithmic singularity has a coefficient of the order $O(\varepsilon^5)$ and, hence, has no effect on the expansion (2.12).

Second, in the range of joining $\sqrt{\varepsilon}\,d_0/2\leqslant|\boldsymbol{x}-\boldsymbol{x}^0|\leqslant\sqrt{\varepsilon}\,d_0$ [or $d_0/(2\sqrt{\varepsilon})\leqslant|\boldsymbol{\xi}|\leqslant d_0/\sqrt{\varepsilon}$ in the extended coordinates], the relation $v(\boldsymbol{x})-w(\varepsilon;\varepsilon^{-1}(\boldsymbol{x}-\boldsymbol{x}^0))=O(\varepsilon^2\sqrt{\varepsilon})$ is satisfied as $\varepsilon\to0$ provided the condition

$$\varepsilon P^*\gamma(\boldsymbol{x}^0;\boldsymbol{x}^0)+\varepsilon^2\sum_{i=1}^{2}M_i^*\gamma^{(i)}(\boldsymbol{x}^0;\boldsymbol{x}^0)=\varepsilon\delta_0^*+\varepsilon\sum_{i=1}^{2}\frac{1}{2R_i^*}(\xi_i^c)^2] \tag{4.2}$$

holds. With allowance for (1.6), (2.7), (3.5), and (3.7), Eq. (4.2) relating the force acting on the punch P to its displacement δ_0 finally becomes

$$P\gamma(\boldsymbol{x}^0;\boldsymbol{x}^0)-P^2k(R_1,R_2;\boldsymbol{x}^0)=\delta_0,$$
$$k(R_1,R_2;\boldsymbol{x}^0)=\sum_{i=1}^{2}R_{3-i}\gamma_i(\boldsymbol{x}^0)\Big[\gamma^{(i)}(\boldsymbol{x}^0;\boldsymbol{x}^0)+\frac12\gamma_i(\boldsymbol{x}^0)\Big]. \tag{4.3}$$

A relation inverse to Eq. (4.3) can be written with the same accuracy in the form

$$P=\frac{\delta_0}{\gamma(\boldsymbol{x}^0;\boldsymbol{x}^0)}+\frac{\delta_0^2}{\gamma(\boldsymbol{x}^0;\boldsymbol{x}^0)^3}k(R_1,R_2;\boldsymbol{x}^0). \tag{4.4}$$

If $\gamma_{12}(\boldsymbol{x}^0)=\gamma_{21}(\boldsymbol{x}^0)\neq0$, formulas (3.5), (3.8) and (3.10), (3.11) for the coordinates of the contact region and its dimensions, respectively, expressions (3.7) for the moments acting on the punch, and the force–displacement relations (4.2)–(4.4) are valid. In this case, the elliptic contact region is rotated about the coordinate axes through a certain angle φ. If $R_1^*=R_2^*$, φ is determined by the quadratic form $\sum_{i,j=1}^{2}\gamma_{ij}(\boldsymbol{x}^0)\xi_i\xi_j$. If, for example, $R_1^*>R_2^*$, we obtain

$$\varphi=-\varepsilon^2\frac{2R_1^*R_2^*}{R_1^*-R_2^*}P^*\gamma_{12}(\boldsymbol{x}^0) \tag{4.5}$$

with accuracy to terms of the order ε^3 [formula (3.4) was obtained with the same accuracy].

By virtue of (2.8), (3.10), and (3.11), the elliptic contact spot has the semiaxes $c(1+m)$ and $c(1-m)$ in the real coordinates. Moreover,

$$c = r_0 \exp\Big\{ -\frac{4\pi D}{P}\,\frac{R_1+R_2}{2R_1R_2} - 4\pi D[\gamma_{11}(\boldsymbol{x}^0)+\gamma_{22}(\boldsymbol{x}^0)] - 1\Big\}, \tag{4.6}$$

$$m = \frac{8\pi D}{P}\,\frac{R_1-R_2}{2R_1R_2} + 8\pi D[\gamma_{22}(\boldsymbol{x}^0)-\gamma_{11}(\boldsymbol{x}^0)]. \tag{4.7}$$

Formulas (4.6) and (4.7) generalize the results obtained by Galin [7]. For a clamped circular plate, we have $\gamma_{11}(0) = \gamma_{22}(0) = -(16\pi D)^{-1}$, and relations (4.6) and (4.7) coincide with the formulas of [7].

It should be noted that the clamped case is considered here only for simplicity. For example, for a simply supported plate, we have $\gamma_{11}(0) = \gamma_{22}(0) = -(16\pi D)^{-1}(3+\nu)(1+\nu)^{-1}$, where ν is Poisson's ratio.

Conclusions. Further complication of the asymptotic representations (see Sec. 4) leads to the fact that the shape of the region becomes nonelliptic. Asymptotic formulas for the contact region were studied in [5, 15].

It is noteworthy that in the case where $R_1^* \neq R_2^*$, the quantity m [see formula (3.11)] is not limited as ε decreases; however, by its geometrical meaning, its absolute value must not exceed unity. This paradox can be explained by the fact that as the difference between the radii of curvature R_1 and R_2 increases, the elongated narrow elliptic contact region becomes a segment. It is noteworthy that the problem of an infinite elastic plate clamped along a line was discussed in [9, § 8.7].

It follows from formulas (3.8) and (4.5)–(4.7) that the parameters of the contact region depend on the dimensions and shape of the plate and the position of the punch center.

The author is grateful to S. A. Nazarov for useful discussions.

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